ELSEVIER

Stochastic Processes and their Applications 64 (1996) 93–102

stochastic processes and their applications

Improved criteria for distributional convergence of point processes

Olav Kallenberg*,[1]

Department of Mathematics, 228 Parker Hall, Auburn University, Auburn, AL 36849-5310, USA

Received February 1995; revised April 1996

Abstract

In earlier work by the author, the convergence in distribution of a sequence of point processes towards a simple limit has been characterized by three conditions, where the first ensures convergence of the supports, the second that the limit of every convergent subsequence is simple, and the third that the sequence is tight. The purpose of this note is to strengthen the mentioned result, by showing that the tightness follows from the other two conditions. Similar results are obtained for convergence of random measures towards a diffuse limit and of row-sums in null-arrays towards an infinitely divisible limit.

Keywords: Random measures and sets; Weak convergence; Infinite divisibility; Null-arrays

AMS classifications: Primary 60G55, 60G57; Secondary 60B10, 60E07

1. Introduction and main results

Throughout this paper, S will denote a locally compact second countable Hausdorff space, and we shall write $\mathscr{G}$, $\mathscr{F}$, and $\mathscr{K}$ for the classes of open, closed, and compact subsets of S, respectively. Furthermore, $\mathscr{B}$ will denote the class of relatively compact Borel sets in S, and $C_K^+ = C_K^+(S)$ the class of continuous functions $f : S \to \mathbb{R}_+$ with compact support.

As in Kallenberg (1975–1986), we define a *random measure* on S to be a locally finite kernel ξ from the basic probability space Ω into S. Thus $\xi_\omega B$ is a locally finite measure in $B \in \mathscr{B}$ for each $\omega \in \Omega$ and a non-negative random variable in $\omega \in \Omega$ for each $B \in \mathscr{B}$. By a *point process* on S is meant an integer valued random measure. The support Ξ of a point process ξ is a *closed random set* in S, in the sense that $\{\Xi \cap B = \emptyset\} = \{\omega \in \Omega;\, \Xi_\omega \cap B = \emptyset\}$ is a measurable subset of Ω for every $B \in \mathscr{B}$. The point process ξ is said to be *simple* if $\sup_s \xi\{s\} \leqslant 1$, i.e., if ξB equals the cardinality of $\Xi \cap B$ for every $B \in \mathscr{B}$.

*E-mail: olavk@mail.auburn.edu.

[1]Research supported by NSF Grant DMS-9303050.

PII S0304-4149(96)00077-4

Random measures on S may be regarded as random elements in the space $\mathscr{M} = \mathscr{M}(S)$ of locally finite measures on S, endowed with the *vague topology* generated by the maps $\mu \mapsto \mu f = \int f\,d\mu$, $f \in C_K^+$. Similarly, we may consider any closed random set in S as a random element in $\mathscr{F}$, equipped with the *Fell topology* generated by the sets $\{F \in \mathscr{F};\ F \cap G \neq \emptyset\}$ and $\{F \in \mathscr{F};\ F \cap K = \emptyset\}$ for arbitrary $G \in \mathscr{G}$ and $K \in \mathscr{K}$. With the latter topology $\mathscr{F}$ becomes a compact second countable Hausdorff space, and the sets $\{F \in \mathscr{F};\ F \cap B = \emptyset\}$ become universally Borel measurable (cf. Matheron (1975, pp. 3, 30)). Convergence in the vague and Fell topologies will be denoted by $\xrightarrow{v}$ and $\xrightarrow{f}$, respectively.

Now let ξ and $\xi_1, \xi_2, \ldots$ be point processes on S, and assume ξ to be simple. For suitable classes $\mathscr{U}, \mathscr{I} \subset \mathscr{B}_\xi = \{B \in \mathscr{B};\ \xi\partial B = 0 \text{ a.s.}\}$, we shall consider the conditions

$$\lim_{n\to\infty} P\{\xi_n U = 0\} = P\{\xi U = 0\}, \quad U \in \mathscr{U}, \tag{1}$$

$$\limsup_{n\to\infty} P\{\xi_n I > 1\} \leqslant P\{\xi I > 1\}, \quad I \in \mathscr{I}. \tag{2}$$

In Kallenberg (1973) (cf. Kallenberg, 1975–1986, Theorem 4.7) it was shown that $\xi_n \xrightarrow{d} \xi$ (i.e., ξ_n converges in distribution to ξ with respect to the vague topology in $\mathscr{M}$), iff (1) and (2) hold for sufficiently rich classes $\mathscr{U}$ and $\mathscr{I}$, and the sequence (ξ_n) is tight. Recall that the latter condition is equivalent to the tightness of $(\xi_n B)$ for every $B \in \mathscr{B}$.

Though this earlier proof involves some technical subtleties, the basic idea is straightforward and shows clearly the role of the three conditions. Thus the tightness implies that (ξ_n) is relatively compact in distribution, so every subsequence contains a further subsequence that converges in distribution towards some limit η. For $U, I \in \mathscr{B}_\eta$ the probabilities in (1) and (2) will then converge to the corresponding limits, which leads to the conditions

$$P\{\eta U = 0\} = P\{\xi U = 0\}, \quad U \in \mathscr{U} \cap \mathscr{B}_\eta, \tag{3}$$

$$P\{\eta I > 1\} \leqslant P\{\xi I > 1\}, \quad I \in \mathscr{I} \cap \mathscr{B}_\eta. \tag{4}$$

If $\mathscr{U}$ is rich enough, then (3) implies $\xi \stackrel{d}{=} \eta^*$, where η^* denotes the counting measure on $\operatorname{supp} \eta$. In particular (4) remains true with ξ replaced by η^*, so if even $\mathscr{I}$ is rich enough we may conclude that $\eta = \eta^*$ a.s. But then $\xi \stackrel{d}{=} \eta$, and $\xi_n \xrightarrow{d} \xi$ follows since the original subsequence was arbitrary.

The tightness condition is obviously crucial for the quoted argument, so it may be surprising to see how conditions (1) and (2) alone are sufficient to ensure the convergence $\xi_n \xrightarrow{d} \xi$, as shown in this paper. Similar improvements are shown to be possible in the related Theorems 4.8, 7.7, and 7.9 of Kallenberg (1975–1986). Some further terminology will be needed, before we can give the precise statements.

Following Norberg (1984), we shall say that a class $\mathscr{U} \subset \mathscr{B}$ is *separating*, if for any $K \in \mathscr{K}$ and $G \in \mathscr{G}$ with $K \subset G$ there exists some $U \in \mathscr{U}$ with $K \subset U \subset G$. We shall further say that $\mathscr{I} \subset \mathscr{B}$ is *pre-separating*, if the class of all finite unions of $\mathscr{I}$-sets is separating. The notions of separating and pre-separating classes are more general and convenient than those of *DC-ring* and *DC-semiring*, used systematically in Kallenberg (1975–1986). In particular we note that, when S is an open subset of a Euclidean space, one may choose $\mathscr{I}$ to be the class of all rectangular boxes in S, and let $\mathscr{U}$ consist of all finite unions of such rectangles. With Norberg we may further note that any [pre-]separating class contains a countable [pre-]separating subclass. Still weaker is the notion of a *covering class* $\mathscr{C}$, which by definition is such that any compact set is covered by finitely many sets from $\mathscr{C}$.

We are now ready to state our improved version of Theorem 4.7 in Kallenberg (1975–1986).

Theorem 1. *Let ξ and $\xi_1, \xi_2, \ldots$ be point processes on S, and assume that ξ is simple. Fix a separating class $\mathscr{U} \subset \mathscr{B}$ and a pre-separating class $\mathscr{I} \subset \mathscr{B}_\xi$. Then $\xi_n \xrightarrow{d} \xi$, whenever*

(i) $\lim_{n\to\infty} P\{\xi_n U = 0\} = P\{\xi U = 0\}$, $U \in \mathscr{U}$,

(ii) $\limsup_{n\to\infty} P\{\xi_n I > 1\} \leqslant P\{\xi I > 1\}$, $I \in \mathscr{I}$.

Note that if we require $\mathscr{U} \subset \mathscr{B}_\xi$, then the two conditions become necessary and sufficient for the convergence $\xi_n \xrightarrow{d} \xi$. A similar remark applies to all subsequent theorems.

The next result gives a similar improvement of Theorem 4.8 in Kallenberg (1975–1986).

Theorem 2. *Let ξ and $\xi_1, \xi_2, \ldots$ be point processes (or random measures) on S, and assume that ξ is simple (or diffuse). Fix some constants $t > s > 0$, a separating class $\mathscr{U} \subset \mathscr{B}$, and a covering class $\mathscr{C} \subset \mathscr{B}_\xi$. Then $\xi_n \xrightarrow{d} \xi$, whenever*

(i) $\lim_{n\to\infty} E\mathrm{e}^{-t\xi_n U} = E\mathrm{e}^{-t\xi U}$, $U \in \mathscr{U}$,

(ii) $\liminf_{n\to\infty} E\mathrm{e}^{-s\xi_n C} \geqslant E\mathrm{e}^{-s\xi C}$, $C \in \mathscr{C}$.

To state the next two results, recall that the random measures ξ_{nj} are said to form a *null-array*, if they are independent for each n and such that

$$\lim_{n\to\infty} \sup_j E[\xi_{nj} B \wedge 1] = 0, \quad B \in \mathscr{B}.$$

Informally this means that $\xi_{nj} B \xrightarrow{P} 0$ as $n \to \infty$, uniformly in j. For point processes we may clearly replace $E[\xi_{nj} B \wedge 1]$ by $P\{\xi_{nj} B > 0\}$.

Just as for random variables any distributional limit ξ of the row-sums $\sum_j \xi_{nj}$ is infinitely divisible, and the law of ξ is given by

$$-\log E\mathrm{e}^{-\xi f} = \alpha f + \int (1 - \mathrm{e}^{-\mu f}) \nu(\mathrm{d}\mu), \quad f \in C_K^+.$$

Here $\alpha \in \mathscr{M}$, and ν is a measure on $\mathscr{M} \setminus \{0\}$ with $\int (\mu B \wedge 1) \nu(\mathrm{d}\mu) < \infty$ for all $B \in \mathscr{B}$. In the point process case we have $\alpha = 0$, and ν is restricted to $\mathscr{N} \setminus \{0\}$, where $\mathscr{N}$ denotes the closed subspace of integer-valued measures in $\mathscr{M}$.

We shall write $I(\alpha, \nu)$ or $I(\nu)$ for the infinitely divisible distribution characterized by (α, ν) or ν, respectively. Let us further write $\mathscr{N}_s$ and $\mathscr{M}_d$ for the classes of simple measures in $\mathscr{N}$ and diffuse measures in $\mathscr{M}$, respectively. Using this notation, we may now state our improved version of Theorem 7.7 in Kallenberg (1975–1986).

Theorem 3. *Let (ξ_{nj}) be a null-array of point processes on S, and let ξ be $I(\nu)$ with $\nu\mathscr{N}_s^c = 0$. Fix a separating class $\mathscr{U} \subset \mathscr{B}$ and a pre-separating class $\mathscr{I} \subset \mathscr{B}_\xi$. Then $\sum_j \xi_{nj} \xrightarrow{d} \xi$, whenever*
(i) $\lim_{n\to\infty} \sum_j P\{\xi_{nj}U > 0\} = \nu\{\mu;\ \mu U > 0\},\ U \in \mathscr{U}$,
(ii) $\limsup_{n\to\infty} \sum_j P\{\xi_{nj}I > 1\} \leqslant \nu\{\mu;\ \mu I > 1\},\ I \in \mathscr{I}$.

We may finally state our improved version of Theorem 7.9 in Kallenberg (1975–1986).

Theorem 4. *Let (ξ_{nj}) be a null-array of point processes (or random measures) on S, and let ξ be $I(\alpha, \nu)$ with $\alpha = \nu\mathscr{N}_s^c = 0$ (or $\nu\mathscr{M}_d^c = 0$). Fix some constants $t > s > 0$, a separating class $\mathscr{U} \subset \mathscr{B}$, and a covering class $\mathscr{C} \subset \mathscr{B}_\xi$. Then $\sum_j \xi_{nj} \xrightarrow{d} \xi$, whenever*
(i) $\lim_{n\to\infty} \sum_j (1 - E\mathrm{e}^{-t\xi_{nj}U}) = t\alpha U + \int (1 - \mathrm{e}^{-t\mu U})\nu(\mathrm{d}\mu),\ U \in \mathscr{U}$,
(ii) $\limsup_{n\to\infty} \sum_j (1 - E\mathrm{e}^{-s\xi_{nj}C}) \leqslant s\alpha C + \int (1 - \mathrm{e}^{-s\mu C})\nu(\mathrm{d}\mu),\ C \in \mathscr{C}$.

Our proof of Theorem 1 is based on the fact that, by Theorem 2.1 in Norberg (1984), condition (i) implies $\operatorname{supp} \xi_n \xrightarrow{d} \operatorname{supp} \xi$ (i.e., $\operatorname{supp} \xi_n$ converges in distribution towards $\operatorname{supp} \xi$ with respect to the Fell topology). By a Skorohod type coupling we may then assume that $\operatorname{supp} \xi_n \xrightarrow{f} \operatorname{supp} \xi$ a.s. Finally we may imitate the proof of Lemma 2.7 in Kallenberg (1975–1986) to see from (ii) that no clustering of point masses is possible.

The other three results are essentially reduced, by various arguments, to the setting of Theorem 1. In particular, we shall use an idea first employed by J. Grandell to derive the diffuse case of Theorem 2 from the corresponding point process result by means of a Cox transformation. The same argument yields a similar improvement of Theorem 3 in Grandell (1976, p. 68).

We may finally remark that convergence criteria related to Theorem 1 have been used extensively in extreme value theory. (See, e.g., Theorem A.1 in Leadbetter et al. (1983, p. 309), and Proposition 3.22 in Resnick (1987, pp. 156f). For many purposes one may replace (ii) by the simpler but not necessary condition $E\xi_n \xrightarrow{v} E\xi$. (This elementary but extremely useful observation is due to T. Kurtz.) With the present improvements, one may hope that even the general criteria will be simple enough to be useful for applications.

2. Some auxiliary results

To prepare for the main proofs, we begin with two elementary results about separating and pre-separating classes.

Lemma 1. *Fix a separating (or pre-separating) class $\mathscr{U} \subset \mathscr{B}$ and a set $G \in \mathscr{G}$. Then the class $\mathscr{U}_G = \{U \in \mathscr{U};\ \bar{U} \subset G\}$ is separating (or pre-separating) in G endowed with the relative topology.*

Proof. First assume that $\mathscr{U}$ is separating. If $K \subset H \subset G$ with $K \in \mathscr{K}$ and $H \in \mathscr{G}$, we may choose $B \in \mathscr{G}$ with $K \subset B \subset \bar{B} \subset H$, and then $U \in \mathscr{U}$ with $K \subset U \subset B$. But then $\bar{U} \subset \bar{B} \subset G$, so $U \in \mathscr{U}_G$.

If $\mathscr{I}$ is pre-separating, the previous case applies to the class $\mathscr{U}$ of finite unions of $\mathscr{I}$-sets, so $\mathscr{U}_G$ is separating in G. It remains to notice that each set in $\mathscr{U}_G$ is a finite union of sets in $\mathscr{I}_G$. □

Lemma 2. *Let $\mathscr{I} \subset \mathscr{B}$ be pre-separating, and fix a set $K \in \mathscr{K}$ covered by some $G_1, \ldots, G_m \in \mathscr{G}$. Then K may also be covered by finitely many sets $I_1, \ldots, I_n \in \mathscr{I}$, such that each I_k lies in some G_j.*

Proof. Let $\mathscr{U}$ denote the class of all finite unions of sets in $\mathscr{I}$. Then $\mathscr{U}$ is separating, so for each $s \in K$ we may choose some $U_s \in \mathscr{U}$, such that $s \in U_s^\circ \subset U_s \subset G_j$ for some j. Since K is compact, it remains covered by finitely many of the sets U_s°. Writing each of the corresponding sets U_s as a finite union of sets in $\mathscr{I}$, we get the desired covering of K by $\mathscr{I}$-sets. □

The next result is based on a simple observation in Norberg (1984). Say that a monotone function $\varphi : \mathscr{B} \to \mathbb{R}$ is *continuous*, if $\varphi(B_n) \to \varphi(B)$ as $B_n \uparrow B$ or $B_n \downarrow B$. Write $\mathscr{B}_\varphi = \{B \in \mathscr{B};\ \varphi(B^\circ) = \varphi(\bar{B})\}$.

Lemma 3. *Let φ and $\varphi_1, \varphi_2, \ldots$ be monotone functions on $\mathscr{B}$, such that $\varphi_n \to \varphi$ on some separating class $\mathscr{U} \subset \mathscr{B}$, and assume that φ is continuous. Then the convergence extends to $\mathscr{B}_\varphi$.*

Proof. Fix any $B \in \mathscr{B}_\varphi$, and let $U, V \in \mathscr{U}$ with $U \subset B^\circ \subset \bar{B} \subset V$. Assuming φ and the φ_n to be non-decreasing, we get

$$\begin{aligned}\varphi(U) = \lim_{n\to\infty} \varphi_n(U) &\leqslant \liminf_{n\to\infty} \varphi_n(B) \\ &\leqslant \limsup_{n\to\infty} \varphi_n(B) \leqslant \lim_{n\to\infty} \varphi_n(V) = \varphi(V).\end{aligned}$$

Now let $U \uparrow B^\circ$ and $V \downarrow \bar{B}$, and conclude by the hypotheses on φ and B that

$$\varphi(B) = \varphi(B^\circ) \leqslant \liminf_{n\to\infty} \varphi_n(B) \leqslant \limsup_{n\to\infty} \varphi_n(B) \leqslant \varphi(\bar{B}) = \varphi(B). \qquad \square$$

The next result relates the vague and Fell topologies for measures in $\mathscr{N}$ and their supports.

Lemma 4. *Let $\mu_1, \mu_2, \ldots \in \mathscr{N}$ and $\mu \in \mathscr{N}_s$ with $\operatorname{supp} \mu_n \xrightarrow{f} \operatorname{supp} \mu$. Then*

$$\limsup_{n\to\infty} (\mu_n B \wedge 1) \leqslant \mu B \leqslant \liminf_{n\to\infty} \mu_n B, \quad B \in \mathscr{B}_\mu.$$

Proof. To prove the left inequality we may assume that $\mu B = 0$. Since $B \in \mathscr{B}_\mu$ we have even $\mu\bar{B} = 0$, so $(\operatorname{supp}\mu) \cap \bar{B} = \emptyset$. By the convergence of the supports we get $(\operatorname{supp}\mu_n) \cap \bar{B} = \emptyset$ for large enough n, which implies

$$\limsup_{n\to\infty}(\mu_n B \wedge 1) \leqslant \limsup_{n\to\infty} \mu_n \bar{B} = 0 = \mu B.$$

To prove the right inequality we may assume that $\mu B = m > 0$. Since $\mathscr{B}_\mu$ is a separating ring, we may choose a partition $B_1, \dots, B_m \in \mathscr{B}_\mu$ of B, such that $\mu B_k = 1$ for each k. Then also $\mu B_k^\circ = 1$ for each k, so $(\operatorname{supp}\mu) \cap B_k^\circ \neq \emptyset$, and by the convergence of the supports we get $(\operatorname{supp}\mu_n) \cap B_k^\circ \neq \emptyset$ for large enough n. Hence

$$1 \leqslant \liminf_{n\to\infty} \mu_n B_k^\circ \leqslant \liminf_{n\to\infty} \mu_n B_k,$$

so

$$\mu B = m \leqslant \sum_{k\leqslant m} \liminf_{n\to\infty} \mu_n B_k \leqslant \liminf_{n\to\infty} \sum_{k\leqslant m} \mu_n B_k = \liminf_{n\to\infty} \mu_n B. \qquad \square$$

We proceed with a useful extension of the classical Skorohod (1956) coupling theorem.

Lemma 5. *Fix some measurable maps $f, f_1, f_2, \dots$ between two Polish spaces S and T, and let $\xi, \xi_1, \xi_2, \dots$ be random elements in S satisfying $f_n(\xi_n) \xrightarrow{d} f(\xi)$. Then there exist, on a suitable probability space, some random elements $\eta \stackrel{d}{=} \xi$ and $\eta_n \stackrel{d}{=} \xi_n$ in S, such that $f_n(\eta_n) \to f(\eta)$ a.s.*

Proof. By Skorohod's theorem there exists a probability space with some random elements $\zeta \stackrel{d}{=} f(\xi)$ and $\zeta_n \stackrel{d}{=} f_n(\xi_n)$ in T, such that $\zeta_n \to \zeta$ a.s. By Lemma 1.1 in Kallenberg (1988) we may further construct, on a possible extension of the latter probability space, some random elements $\eta \stackrel{d}{=} \xi$ and $\eta_n \stackrel{d}{=} \xi_n$ in S, such that $\zeta = f(\eta)$ a.s. and $\zeta_n = f_n(\eta_n)$ a.s. for each n. $\square$

The last result will now be applied to our point processes and their supports.

Lemma 6. *Let ξ and $\xi_1, \xi_2, \dots$ be point processes on S, and assume for some separating class $\mathscr{U} \subset \mathscr{B}$ that*

$$\lim_{n\to\infty} P\{\xi_n U = 0\} = P\{\xi U = 0\}, \quad U \in \mathscr{U}. \tag{5}$$

Then there exist, on a suitable probability space, some point processes $\eta \stackrel{d}{=} \xi$ and $\eta_n \stackrel{d}{=} \xi_n$ on S with $\operatorname{supp}\eta_n \xrightarrow{f} \operatorname{supp}\eta$ a.s.

Proof. By Theorem 2.1 of Norberg (1984), relation (5) implies $\operatorname{supp}\xi_n \xrightarrow{d} \operatorname{supp}\xi$. Since $\mathscr{N}$ and $\mathscr{F}$ are Polish and the mapping $\mu \mapsto \operatorname{supp}\mu$ is measurable, the assertion now follows by Lemma 5. $\square$

3. Proofs of the main results

The results of the previous section will now be used to prove the four main theorems stated in Section 1.

Proof of Theorem 1. By (i) and Lemma 6 we may assume that $\operatorname{supp}\xi_n \xrightarrow{f} \operatorname{supp}\xi$ a.s. and since ξ is simple we get by Lemma 4,

$$\limsup_{n\to\infty}(\xi_n B \wedge 1) \leqslant \xi B \leqslant \liminf_{n\to\infty} \xi_n B \quad \text{a.s.}, \qquad B \in \mathscr{B}_\xi. \tag{6}$$

Next we note that, for m and n restricted to $\mathbb{Z}_+$,

$$\begin{aligned}\{n \leqslant m \leqslant 1\}^c &= \{m > 1\} \cup \{m < n \wedge 2\}\\ &= \{n > 1\} \cup \{m = 0, n = 1\} \cup \{m > 1 \geqslant n\},\end{aligned}$$

where all unions are disjoint. Substituting $m = \xi I$ and $n = \xi_n I$, we get by (ii) and (6),

$$\lim_{n\to\infty} P\{\xi I < \xi_n I \wedge 2\} = 0, \quad I \in \mathscr{I}. \tag{7}$$

For any sets $B \subset I \in \mathscr{I}$ we get with $B' = I \setminus B$,

$$\begin{aligned}\{\xi_n B > \xi B\} &\subset \{\xi_n I > \xi I\} \cup \{\xi_n B' < \xi B'\}\\ &\subset \{\xi_n I \wedge 2 > \xi I\} \cup \{\xi I > 1\} \cup \{\xi_n B' < \xi B'\}.\end{aligned} \tag{8}$$

If, instead, $B \in \mathscr{B}_\xi$ is covered by $I_1, \ldots, I_m \in \mathscr{I}$, it may be partitioned into subsets $B_k \in \mathscr{B}_\xi \cap I_k$, $k = 1, \ldots, m$, and we get by (6)–(8),

$$\limsup_{n\to\infty} P\{\xi_n B > \xi B\} \leqslant P \bigcup_k \{\xi I_k > 1\}. \tag{9}$$

Now fix any $B \in \mathscr{B}_\xi$ and $K \in \mathscr{K}$ with $\bar{B} \subset K^\circ$, a metric d in S, and a constant $\varepsilon > 0$. By Lemma 2 we may choose $I_1, \ldots, I_m \in \mathscr{I}$ with d-diameter $< \varepsilon$ such that $B \subset \bigcup_k I_k \subset K$. Since ξ is simple, the right-hand side of (9) is bounded by $P\{\rho_K < \varepsilon\}$, where ρ_K denotes the minimum d-distance between points in $(\operatorname{supp}\xi) \cap K$. Now $\rho_K > 0$ a.s., and since $\varepsilon > 0$ is arbitrary we get $P\{\xi_n B > \xi B\} \to 0$. Combining this with the second relation in (6) yields $\xi_n B \xrightarrow{P} \xi B$. In particular

$$(\xi_n B_1, \ldots, \xi_n B_k) \xrightarrow{d} (\xi B_1, \ldots, \xi B_k), \quad B_1, \ldots, B_k \in \mathscr{B}_\xi,\ k \in \mathbb{N},$$

so $\xi_n \xrightarrow{d} \xi$ by Theorem 4.2 in Kallenberg (1975–1986). □

Proof of Theorem 2. In the point process case we define

$$p = 1 - \mathrm{e}^{-t}, \qquad r = -\log\left\{1 - \frac{1 - \mathrm{e}^{-s}}{1 - \mathrm{e}^{-t}}\right\},$$

let η and $\eta_1, \eta_2, \ldots,$ be p-thinnings of ξ and $\xi_1, \xi_2, \ldots$, respectively (cf. Kallenberg (1975–1986, p. 16)), and note that (i) and (ii) are equivalent to

$$\lim_{n\to\infty} P\{\eta_n U = 0\} = P\{\eta U = 0\}, \quad U \in \mathscr{U}, \tag{10}$$

$$\liminf_{n\to\infty} E\mathrm{e}^{-r\eta_n C} \geqslant E\mathrm{e}^{-r\eta C}, \quad C \in \mathscr{C}. \tag{11}$$

By (10) and Lemma 6 we may assume that $\operatorname{supp}\eta_n \to \operatorname{supp}\eta$ a.s., so that by Lemma 4,

$$\liminf_{n\to\infty} \eta_n B \geqslant \eta B \quad \text{a.s.,} \quad B \in \mathscr{B}_\eta = \mathscr{B}_\xi. \tag{12}$$

Now fix any $C \in \mathscr{C}$, and write

$$\begin{aligned}0 &\leqslant (1-\mathrm{e}^{-r})E[\mathrm{e}^{-r\eta C};\ \eta_n C > \eta C]\\ &\leqslant E[\mathrm{e}^{-r\eta C} - \mathrm{e}^{-r\eta_n C};\ \eta_n C > \eta C]\\ &= E\mathrm{e}^{-r\eta C} - E\mathrm{e}^{-r\eta_n C} - E[\mathrm{e}^{-r\eta C} - \mathrm{e}^{-r\eta_n C};\ \eta_n C < \eta C].\end{aligned}$$

By (11) and (12) we obtain $E[\mathrm{e}^{-r\eta C};\ \eta_n C > \eta C] \to 0$, and since $\mathrm{e}^{-r\eta C} > 0$ it follows that

$$\lim_{n\to\infty} P\{\eta_n C > \eta C\} = 0, \quad C \in \mathscr{C}. \tag{13}$$

Any $B \in \mathscr{B}_\eta$ may be covered by finitely many sets $C_1,\ldots,C_m \in \mathscr{C}$, and from (12) and (13) we may conclude as in (9) that

$$P\{\eta_n B > \eta B\} \leqslant \sum_k P\{\eta_n C_k > \eta C_k\} \to 0.$$

Combining this with (12) yields $\eta_n B \xrightarrow{P} \eta B$ for all $B \in \mathscr{B}_\eta$, and $\eta_n \xrightarrow{d} \eta$ follows as before. Finally $\xi_n \xrightarrow{d} \xi$ by Exercise 4.5 in Kallenberg (1975–1986).

In the random measure case we may take η and $\eta_1, \eta_2, \ldots$ to be Cox processes directed by $t\xi$ and $t\xi_1, t\xi_2, \ldots$, respectively. Then (10) and (11) remain true with $r = -\log(1 - s/t)$, and we may argue as before. $\square$

Proof of Theorem 3. First assume that

$$\sum_j P\{\xi_{nj} \neq 0\} \to \nu\{\mu;\ \mu \neq 0\} < \infty.$$

Let ν_n denote the restriction of $\sum_j P \circ \xi_{nj}^{-1}$ to $\mathscr{N}\setminus\{0\}$. If $\nu \neq 0$, the probability measures $\nu_n/\nu_n(\mathscr{N})$ and $\nu/\nu(\mathscr{N})$ satisfy the conditions in Theorem 1, and we get $\nu_n \xrightarrow{w} \nu$. The same convergence holds trivially when $\nu = 0$. By Theorem 4.2 in Kallenberg (1975–1986) it follows that, for any $B_1,\ldots,B_m \in \mathscr{B}_\xi$, $m \in \mathbb{N}$,

$$\sum_j P \circ (\xi_{nj}B_1,\ldots,\xi_{nj}B_m)^{-1} \xrightarrow{w} \nu \circ (\pi_{B_1},\ldots,\pi_{B_m})^{-1} \quad \text{on } \mathbb{Z}_+^m\setminus\{0\},$$

where π_B denotes the mapping $\mu \mapsto \mu B$. Hence $\sum_j \xi_{nj} \xrightarrow{d} \xi$ by Theorem 6.1 in the same reference.

In the general case, fix any $f \in C_K^+$ and choose a set $G \in \mathscr{B}_\xi \cap \mathscr{G}$ containing the closure of the support of f. From Lemma 3 we note that (i) remains true for any $U \in \mathscr{B}_\xi$, and in particular

$$\sum_j P\{\xi_{nj} G > 0\} \to \nu\{\mu;\ \mu G > 0\} < \infty.$$

By Lemma 1 the class $\mathscr{I}_G = \{I \in \mathscr{I}; \bar{I} \subset G\}$ is pre-separating in G with the relative topology, so the previous case applies to the restrictions $G\xi = \xi(G \cap \cdot)$ and $G\xi_{nj} = \xi_{nj}(G \cap \cdot)$, and we obtain $\sum_j G\xi_{nj} \xrightarrow{d} G\xi$ in $\mathscr{M}(G)$. Noting that even $f \in C_K^+(G)$, we get in particular

$$\sum_j \xi_{nj} f = \sum_j (G\xi_{nj}) f \xrightarrow{d} (G\xi) f = \xi f,$$

and since f was arbitrary, Theorem 4.2 in Kallenberg (1975–1986) yields $\sum_j \xi_{nj} \xrightarrow{d} \xi$. □

Proof of Theorem 4. Beginning with the point process case, we may first assume that ν and the measures $\nu_n = \sum_j P \circ \xi_{nj}^{-1}$ on $\mathscr{N} \setminus \{0\}$ are uniformly bounded. They may then be extended to measures χ and χ_n on $\mathscr{N}$ with the same finite total mass. By Theorem 2 we get $\chi_n \xrightarrow{w} \chi$, and using Theorems 4.2 and 6.1 in Kallenberg (1975–1986) as before, we may conclude that $\sum_j \xi_{nj} \xrightarrow{d} \xi$.

In the general point process case, fix any finite union U of sets $C_1, \ldots, C_m \in \mathscr{C}$, and note that (ii) remains true for the restrictions $U\xi$ and $U\xi_{nj}$, with $\mathscr{C}$ replaced by the class $\{C_1, \ldots, C_m\} \cup (\mathscr{B}_\xi \cap U^c)$. By Lemma 3 the same restrictions satisfy (i) with $\mathscr{U}$ replaced by $\mathscr{B}_\xi$, and in particular

$$\begin{aligned}(1 - \mathrm{e}^{-t}) \sum_j P\{\xi_{nj} U > 0\} &\leqslant E(1 - \mathrm{e}^{-t\xi_{nj} U}) \\ &\to \int (1 - \mathrm{e}^{-t\mu U}) \nu(\mathrm{d}\mu) \\ &\leqslant \nu\{\mu; \mu U > 0\} < \infty.\end{aligned}$$

Hence the previous case yields $\sum_j U\xi_{nj} \xrightarrow{d} U\xi$, and since $\mathscr{C}$ is covering we get $\sum_j \xi_{nj} f \xrightarrow{d} \xi f$ for all $f \in C_K^+$, which means that $\sum_j \xi_{nj} \xrightarrow{d} \xi$.

Turning to the random measure case, we may fix any $u > t$ and let η and η_{nj} be Cox processes directed by $u\xi$ and $u\xi_{nj}$, respectively, where the η_{nj} are assumed to be independent for each n. Fixing any $r \in [0, u)$ and writing $r' = -\log(1 - r/u)$, we get for arbitrary $B \in \mathscr{B}$,

$$E\mathrm{e}^{-r'\eta B} = E\mathrm{e}^{-r\xi B}, \qquad E\mathrm{e}^{-r'\eta_{nj} B} = E\mathrm{e}^{-r\xi_{nj} B}. \tag{14}$$

The process η is again infinitely divisible, and by (14) its Lévy measure ν' satisfies

$$\int (1 - \mathrm{e}^{-r'\mu B}) \nu'(\mathrm{d}\mu) = r\alpha B + \int (1 - \mathrm{e}^{-r\mu B}) \nu(\mathrm{d}\mu). \tag{15}$$

Writing $t' = -\log(1 - t/u)$ and $s' = -\log(1 - s/u)$, it is seen from (14) and (15) that (i) and (ii) remain valid with ξ_{nj}, ν, α, t, and s replaced by η_{nj}, ν', 0, t', and s', respectively. Further, note that, $\nu \mathscr{M}_d^c = 0$ implies $\nu' \mathscr{N}_s^c = 0$. The statement in the point process case now yields $\sum_j \eta_{nj} \xrightarrow{d} \eta$, and it follows as before that $\sum_j \xi_{nj} \xrightarrow{d} \xi$. □

References

J. Grandell, Doubly stochastic Poisson processes, Lecture Notes in Math., Vol. 529 (Springer, Berlin, 1976).

O. Kallenberg, Characterization and convergence of random measures and point processes, Z. Wahrscheinlichkeitstheorie verw. Gebiete 27 (1973) 9–21.

O. Kallenberg, Random Measures, edns. 1–4 (Akademie-Verlag, Berlin, and Academic Press, London, 1975–1986).

O. Kallenberg, Spreading and predictable sampling in exchangeable sequences and processes, Ann. Probab. 16 (1988) 508–534.

M.R. Leadbetter, G. Lindgren and H. Rootzén, Extremes and Related Properties of Random Sequences and Processes (Springer, New York, 1983).

G. Matheron, Random Sets and Integral Geometry (Wiley, New York, 1975).

T. Norberg, Convergence and existence of random set distributions, Ann. Probab. 12 (1984) 726–732.

S.I. Resnick, Extreme Values, Regular Variation, and Point Processes (Springer, New York, 1987).

A.V. Skorohod, Limit theorems for stochastic processes, Theor. Probab. Appl. 1 (1956) 261–290.